

## *Comments on 'Balance and the slow quasimanifold: some explicit results'*

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## NOTES AND CORRESPONDENCE

## Comments on “Balance and the Slow Quasimanifold: Some Explicit Results”

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**1. Introduction**

The concept of slow vortical dynamics and its role in theoretical understanding is central to geophysical fluid dynamics. It leads, for example, to “potential vorticity thinking” (Hoskins et al. 1985). Mathematically, one imagines an invariant manifold within the phase space of solutions, called the slow manifold (Leith 1980; Lorenz 1980), to which the dynamics are constrained. Whether this slow manifold truly exists has been a major subject of inquiry over the past 20 years. It has become clear that an exact slow manifold is an exceptional case, restricted to steady or perhaps temporally periodic flows (Warn 1997). Thus the concept of a “fuzzy slow manifold” (Warn and Ménard 1986) has been suggested. The idea is that nearly slow dynamics will occur in a stochastic layer about the putative slow manifold. The natural question then is, how thick is this layer?

In a recent paper, Ford et al. (2000) argue that Lighthill emission—the spontaneous emission of freely propagating acoustic waves by unsteady vortical flows—is applicable to the problem of balance, with the Mach number  $Ma$  replaced by the Froude number  $F$ , and that it is a fundamental mechanism for this fuzziness. They consider the rotating shallow-water equations and find emission of inertia-gravity waves at  $O(F^2)$ . This is rather surprising at first sight, because several studies of balanced dynamics with the rotating shallow-water equations have gone beyond second order in  $F$ , and found only an exponentially small unbalanced component (Warn and Ménard 1986; Lorenz and Krishnamurthy 1987; Bokhove and Shepherd 1996; Wirosoetisno and Shepherd 2000). We have no technical objection to the analysis of Ford et al. (2000), but wish to point out that it depends crucially on  $R \gtrsim 1$ , where  $R$  is the Rossby number. This condition requires the ratio

of the characteristic length scale of the flow  $L$  to the Rossby deformation radius  $L_R$  to go to zero in the limit  $F \rightarrow 0$ . This is the low Froude number scaling of Charney (1963), which, while originally designed for the Tropics, has been argued to be also relevant to mesoscale dynamics (Riley et al. 1981). If  $L/L_R$  is fixed, however, then  $F \rightarrow 0$  implies  $R \rightarrow 0$ , which is the standard quasigeostrophic scaling of Charney (1948; see, e.g., Pedlosky 1987). In this limit there is reason to expect the fuzziness of the slow manifold to be “exponentially thin,” and balance to be much more accurate than is consistent with (algebraic) Lighthill emission.

**2. Multiscale frequency matching and Lighthill emission**

In the original Lighthill (1952) theory, acoustic waves are emitted from a vortical flow through multiscale frequency matching. Consider the approximate dispersion relations shown in Fig. 1a. [Figure 1 is admittedly only a cartoon, but it captures the essential physics. In reality, the lines correspond to peaks in the frequency-wave-number power spectrum, which is where any  $O(1)$  vortical-acoustic wave coupling would have to occur.] The vortical motion obeys

$$\omega_v \sim Uk, \quad (1)$$

where  $\omega$  is the frequency,  $U$  is a characteristic velocity, and  $k$  is the wavenumber, while the acoustic waves obey

$$\omega_A = ck, \quad (2)$$

where  $c$  is the speed of sound. In the linear problem  $\omega_v = 0$ , so (1) can be regarded as a nonlinear broadening of the zero-frequency linear mode. The Mach number  $Ma \equiv U/c$  is the dimensionless measure of the amplitude of the vortical motion, for a given medium (characterized by  $c$ ); the limit  $Ma \rightarrow 0$  is thus the small-amplitude limit for which  $\omega_v \rightarrow 0$  at fixed  $\omega_A$ . In this limit, the slope of the vortical branch is small compared to that of the acoustic wave branch [and nonlinear broadening does not significantly alter (2)]. This implies a formal

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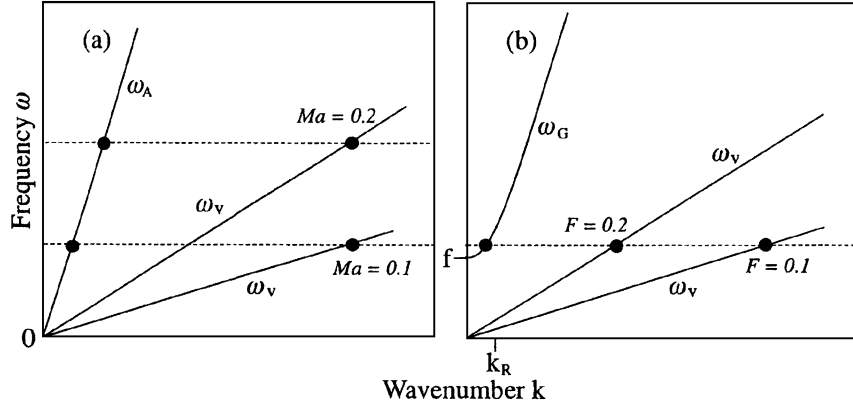


FIG. 1. (a) Multiscale frequency matching in Lighthill emission. The acoustic wave branch is  $\omega_A$ , and  $\omega_v$  is the vortical branch, shown for two values of  $Ma$ . The limit  $Ma \rightarrow 0$  at a fixed vortical length scale  $k_v^{-1}$  always leads to frequency matching and efficient emission of acoustic waves. (b) Multiscale frequency matching in the rotating shallow-water analysis of Ford et al. (2000). The Rossby deformation wavenumber is  $k_R \equiv f/\sqrt{gH}$ ,  $\omega_G$  is the inertia-gravity wave branch, and  $\omega_v$  is the vortical branch, shown for two values of  $F$ . The limit  $F \rightarrow 0$  with  $R \geq 1$  considered by Ford et al. (2000) requires  $k_v \rightarrow \infty$ , as indicated. If  $k_v$  is held fixed, then multiscale frequency matching does not occur in the limit  $F \rightarrow 0$ .

timescale separation in the respective motion like  $Ma^{-1}$ , for a single  $k$ . Since both dispersion curves pass through the origin, we see that vortical motion of a fixed length scale  $k_v^{-1}$  will force acoustic waves, of wavelength  $k_A^{-1}$ , for arbitrarily small  $Ma$ . Matching the frequencies implies  $k_A = Ma k_v$ ; the limit  $Ma \rightarrow 0$  for a given  $k_v$  corresponds to the limit  $k_A \rightarrow 0$  (Fig. 1a).

The rotating case relevant to the problem of atmospheric balance (outside the Tropics) is, however, fundamentally different. Consider the approximate dispersion relations shown in Fig. 1b. As before the vortical motion is a nonlinear broadening of the zero-frequency linear mode and obeys

$$\omega_v \sim Uk, \quad (3)$$

but now the inertia-gravity waves obey the relation

$$\omega_G = \sqrt{f^2 + gHk^2}, \quad (4)$$

where  $f$  is the Coriolis parameter,  $g$  is the gravitational acceleration, and  $H$  is the depth of the fluid at rest. For a given medium (characterized by  $f$ ,  $g$ , and  $H$ ), there are two dimensionless measures of the amplitude of the vortical motion: the Froude number  $F \equiv U/\sqrt{gH}$  and the Rossby number  $R \equiv U/fL = Uk/f$ . If  $F \ll 1$ , as assumed by Ford et al. (2000), then the slope of the vortical branch is small compared to that of the asymptote for the inertia-gravity wave branch. Again we have a formal timescale separation in the respective motion, like  $F^{-1}$ , for a single  $k$ . But if we seek the condition for multiscale frequency matching then, in contrast to the acoustic case, there is a restriction. Similar to the above, let  $k_v$  and  $k_G$  denote the wavenumbers of the vortical and inertia-gravity wave motions, respectively. Frequency matching then requires

$$\sqrt{f^2 + gHk_G^2} \sim Uk_v \Leftrightarrow k_G \sim \sqrt{\frac{U^2k_v^2 - f^2}{gH}}. \quad (5)$$

Since  $k_G \in \Re$  for propagating waves, we must therefore have

$$R = \frac{Uk_v}{f} \geq 1. \quad (6)$$

This condition is crucial to the analysis of Ford et al. (2000) (see the bottom of p. 1239b of their paper), for otherwise the solution of their (41) is evanescent. In the limit  $F \rightarrow 0$ , it is achieved by taking  $k_v/k_R \rightarrow \infty$  (Fig. 1b), where  $k_R \equiv f/\sqrt{gH}$  is the Rossby deformation wavenumber.

The key difference between the acoustic and the rotating shallow-water cases is that, in the latter, both dispersion curves do not pass through the origin. If we fix  $k_v$  and  $k_R$ , that is, fix the dimensionless length scale of the vortical motion, and consider the limit  $F \rightarrow 0$ , then necessarily  $R \rightarrow 0$  as well, since

$$R = \frac{k_v}{k_R} F. \quad (7)$$

This is the small-amplitude limit for which  $\omega_v \rightarrow 0$  at fixed  $\omega_G$ , and would seem to be the analogue to the situation considered in Lighthill emission. But in this limit, multiscale frequency matching cannot occur. The Ford et al. (2000) analysis no longer applies, for now motion generated by the vortical flow with  $\omega_v \sim Uk_v$  will produce inertia-gravity waves with the purely imaginary wavenumber

$$k_G \sim Fk_v \sqrt{1 - \frac{1}{R^2}}. \quad (8)$$

Thus the forced emission at  $O(F^2)$  fails to occur because frequency matching is no longer possible; these waves are trapped, not freely propagating.

We certainly do not argue that inertia-gravity wave emission is nonexistent, only that it is not properly described by Lighthill emission for standard quasigeostrophic scaling where the dimensionless length scale of the source is held fixed in the limiting process. For chaotic slow dynamics there will always be some frequency matching, even for  $R \rightarrow 0$ , because there will inevitably be high-frequency tails to the vortical power spectrum that will overlap the inertia-gravity wave frequencies (Errico 1982). But this fast-slow coupling is much weaker than that resulting from the Lighthill mechanism depicted in Fig. 1a. In contrast with the algebraic emission at  $O(F^2)$  found by Ford et al. (2000), other studies of balance have tended to find exponentially small imbalance of  $O[\exp(-F^{-1})]$  as  $F \rightarrow 0$  (Warn and Ménard 1986; Lorenz and Krishnamurthy 1987; Bokhove and Shepherd 1996; Wirosoetisno and Shepherd 2000). This means that the imbalance is too small to be captured by a power series in  $F$ . It is true that the cited studies all consider severe spectral truncations of the rotating shallow-water equations and therefore cannot describe the multiscale interaction that lies behind both Lighthill emission and the frequency matching considered by Ford et al. (2000). The exponentially small imbalance found in these studies for  $F \rightarrow 0$ ,  $R \geq 1$  is therefore an artifact of the spectral truncation. However, it is clear from the above discussion that the exponentially small imbalance found for  $F \rightarrow 0$ ,  $R \rightarrow 0$  at fixed  $k_v$  (for a given  $k_R \equiv f/\sqrt{gH}$ ) is almost certainly not an artifact of the spectral truncation.

Ford et al. (2000) refer to the Hamiltonian theory of coupled oscillators (Camassa 1995; Bokhove and Shepherd 1996) as part of the motivation for expecting imbalance. This is reasonable, although it should be noted that in the Lorenz (1986) model, *all* pendulum solutions, not just the homoclinic orbit, are slow solutions, and almost all of these survive for small but finite  $F$ . Moreover, the fuzziness of the slow manifold in this case is ensured by the Kolmogorov–Arnold–Moser (KAM) theorem to be exponentially small in  $F$  (Bokhove and Shepherd 1996). The near-integrable case studied by Bokhove and Shepherd (1996) is somewhat restrictive, but Wirosoetisno and Shepherd (2000) studied a model with chaotic slow dynamics and hence a full frequency spectrum that overlaps the fast frequencies. Nevertheless, the growth of the imbalance was shown to be exponentially small in  $F$ . This result suggests that the simple argument behind Fig. 1 is robust to the inclusion of high-frequency tails.

The coupled vortical-gravity wave instability of a circular vortex (Ford 1994), cited by Ford et al. (2000) as supporting evidence, is also consistent with this picture. The asymptotic analysis of Ford (1994) for  $F \ll 1$  requires  $R > 1$ ; in the limit  $F \rightarrow 0$ ,  $R \rightarrow 0$ , the azimuthal wavenumber of the instability goes to infinity and the

growth rate is found to be exponentially small in  $F$  (Ford 1994). (See also Nore and Shepherd 1997, section 7b.) Note that this analysis concerns the full shallow-water equations, not a severe spectral truncation.

The fact that the equations of motion can be rearranged, following Lighthill (1952), into a form with a linear wave operator on the left-hand side, and the nonlinear terms on the right-hand side, does not prove that the nonlinear terms force waves. There is always the possibility that the fast variables are slaved (e.g., Warn et al. 1995). The question is, rather, whether slaving breaks down at some order in a small parameter expansion, or whether it does so beyond all orders. This remains an open question for the continuous equations.

### 3. Summary

The point we wish to emphasize is that Lighthill emission exists in the acoustic wave case because frequency matching occurs for fixed  $k_v$  as  $Ma \rightarrow 0$ , for all  $Ma > 0$ . In contrast, the dispersion curve of the fast motion in the rotating case does not pass through the origin, but instead has a low-frequency cutoff (Fig. 1). This means that for standard quasigeostrophic scaling, with fixed  $k_v$  (and fixed  $k_R$ ), taking  $F \rightarrow 0$  eliminates the direct frequency matching, and hence the algebraic emission provided by the Lighthill mechanism. There is a fuzziness to the quasigeostrophic slow manifold, but it depends on the high-frequency tail of the slow dynamics and could well be exponentially small, and not capturable by a power series expansion.

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